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Characteristics of conic segments in Bézier form

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Abstract

The rational Bézier form has become a standard in CAD-CAM packages and data exchange formats, because it encompasses both conic segments (in the quadratic case) and general free-form geometry. We present several results on the relationship between the quadratic rational Bézier form and the classical definition of conics in terms of their characteristics, such as foci, centre, axis and eccentricity. First, we recall a simple geometric procedure to compute arbitrary conic segments of given focus in Bézier form. Second, from this procedure we derive the geometric characteristics of a given Bézier conic in a straightforward manner, by employing complex arithmetic. For a central conic, a simple quadratic equation defines the foci location, and its solution furnishes not only an explicit formula for the foci, but also for the centre, axis direction and linear eccentricity.

1 Introduction

The original polynomial Bézier model has a shortcoming, namely that it does not encompass all conics (only the parabola). This is a serious shortcoming as, for instance, conics find widespread use in optical and telecommunication instruments [3], owing to their remarkable reflective properties. To remedy this deficiency, the rational Bézier model was developed, where each control point is assigned a weight. Any conic segment can be expressed in quadratic rational Bézier form, and vice versa, any quadratic rational Bézier curve is a conic segment. This rational Bézier form has thus become a standard in CAD-CAM packages and data exchange formats.

The Bézier representation of conics is found in most textbooks on CAD [5],[6],[9], and all NURBS monographs [4],[12],[13]. However, only Piegl and Tiller [12] include formulae, due to Lee [10], for obtaining the geometric characteristics (centre, foci, axes) of an already constructed rational quadratic Bézier segment, in terms of its weights and control points. Recently, Xu et al. [18] have derived explicit formulae, based on Lee's results, for computing the eccentricity of a Bézier conic. Although not difficult, Lee's computations [10] are rather involved. They require the use of Lagrange multipliers to derive the axis length, and involve complex formulae in terms of vector and cross products. Furthermore, no simple direct expressions are given for the foci, the relevant points regarding reflective properties sketched in figs. 1,2.

This shortcoming was somehow tackled by Albrecht [1], who emphasizes on determining the foci of a given conic in Bézier form. However, her derivation requires heavy and sophisticated mathematical machinery, such as computing the singular points of a certain algebraic curve of degree four, or the use of a symbolic algebra package to derive some expressions.

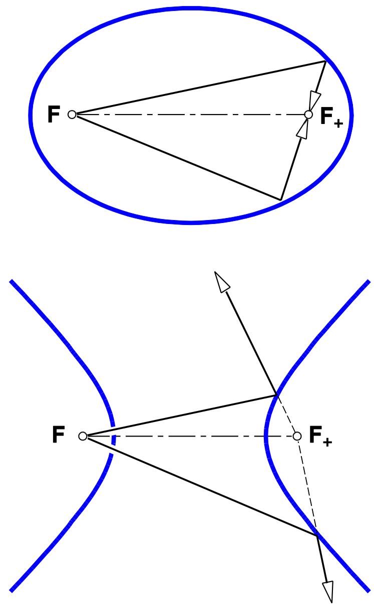

***Fig. 1 Reflective properties of central conics with foci** F_-, F_+*

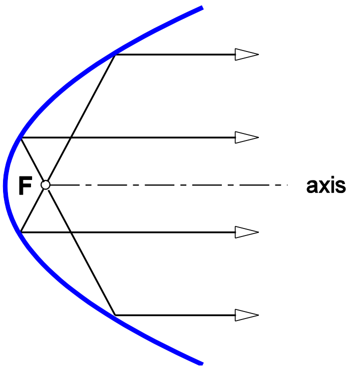

***Fig. 2 Reflective properties of a parabola with focus* F**

Explicit formulae for all geometric characteristic are due to Goldman and Wang [8], although they do not employ the customary Bézier representation. They derive their results in an algebraic way, from the invariants of rational quadratic parameterizations under rational linear reparameterizations.

We advocate an alternative, more geometric approach, based on representing conics in the complex plane $\mathbb{C}$. Complex analysis is a powerful and elegant tool that, though restricted to the planar case, facilitates the construction and analysis of curves in CAGD, as Farouki [7] notes. It might seem that representing a curve in $\mathbb{C}$, instead of in the customary 2D-Euclidean space, could be just a matter of taste. The point is that the space $\mathbb{C}$ enjoys the algebraic structure of field, where not only can points be added, but also multiplied and divided, and square roots become meaningful. As a consequence, complex arithmetic drastically simplifies the expressions for the foci, centre, and linear eccentricity of a Bézier conic.

The paper is arranged as follows. In Section 2, we first characterize the focus in a trivial form with complex notation. This basic result allows us to obtain the foci in Section 3, as the solutions of a (complex) quadratic equation. The centre and linear eccentricity come as by-products. We also sketch how to computer other conic characteristics (Section 4). Finally, conclusions are drawn in Section 5.

2 Characterizing the foci with complex products

Before trying to obtain the foci of a given quadratic Bézier conic, we recall the inverse problem, that is, how to construct arbitrary Bézier conics of given focus **F**. Without loss of generality, we employ the standard form (with unit weights for the endpoints $\mathbf{b}_0$,$\mathbf{b}_2$.). Sánchez-Reyes [16] shows that, whereas we can choose arbitrarily $\mathbf{b}_0$,$\mathbf{b}_2$, the inner point $\mathbf{b}_1$ and weight $w_1=w$ are constrained:

1) The point $\mathbf{b}_1$ lies on the bisector of the lines $\mathbf{F}\mathbf{b}_0$,$\mathbf{F}\mathbf{b}_2$, i. e., so that the segments $\mathbf{b}_0\mathbf{b}_1$ and $\mathbf{b}_1\mathbf{b}_2$ see **F** with the same angle Δ (fig. 3a).
2) The inner weight w takes a specific value, determined by the radial distances r_k:

$$w^2 = \frac{r_0 r_2}{r_1^2}, \qquad r_k = |\mathbf{r}_k|, \quad \mathbf{r}_k = \mathbf{b}_k - \mathbf{F} \, . \tag{1}$$

Condition (1) simply rewrites in Bézier representation a classical result [14], which states that the intersection of any two tangents to a conic and both points of contact are seen from **F** within equal angles Δ.

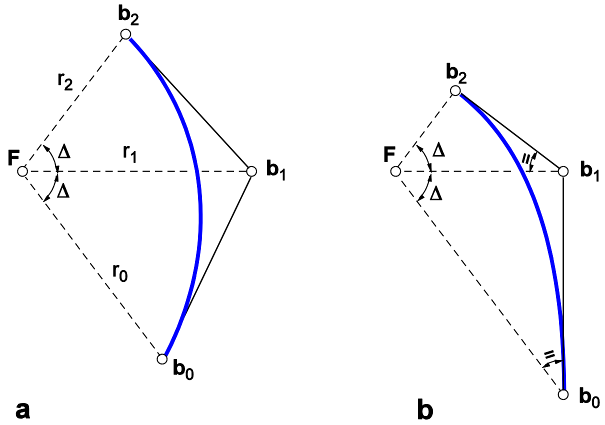

***Fig. 3 Bézier points* $\mathbf{b}_k$ *of a conic with focus* F:**
a) General conic. b) Parabola.

The radial vectors $\mathbf{r}_k$ (1) can be hence written in polar form as complex exponentials [11] of moduli r_k, and arguments θ_k equally spaced by the angle Δ:

$$\mathbf{r}_k = r_k \, e^{i\theta_k}, \qquad \theta_{k+1} - \theta_k = \Delta \, .$$

By introducing these complex exponentials and the value w (1), we obtain a startlingly simple characterization of the focus **F**, in terms of complex products:

$$(w\mathbf{r}_1)^2 = \mathbf{r}_0\mathbf{r}_2, \qquad \mathbf{r}_k = \mathbf{b}_k - \mathbf{F} \, . \tag{2}$$

For the case of a parabola ($w=1$), this relationship indicates that the values r_k form a geometric progression, which admits an intuitive interpretation: the adjacent triangles $\mathbf{F}\mathbf{b}_0\mathbf{b}_1$ and $\mathbf{F}\mathbf{b}_1\mathbf{b}_2$ are similar (fig. 3b). This geometric property was already noted by Sánchez-Reyes [15], and derived also by Ueda [17] from the pedal-point construction of a parabola.

3 Computing the foci, centre and linear eccentricity

3.1 Conic classification

Suppose that we are given a Bézier conic in standard form, of points $\mathbf{b}_k$ and inner weight w_1=w. To find the focus **F**, simply interpret equality (2) as an equation in the unknown **F**, and solve it.

As shown in this Section, simple algebra yields the roots, according to the well-know case distinction (Fig. 4) that determines the conic type: ellipse or hyperbola ($w \neq 1$), and parabola ($w=1$).

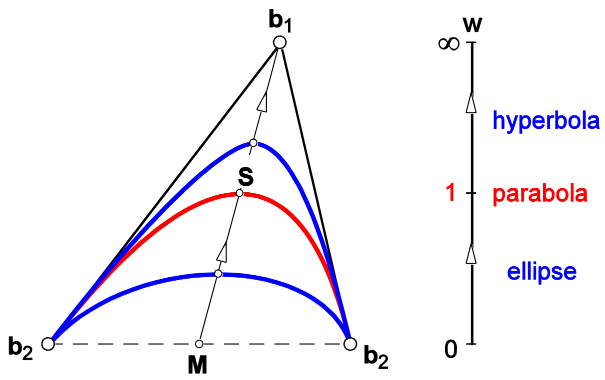

Fig. 4 Conic type accordino to the inner weight *w.*

To fix our ideas, we assume the customary condition $w > 0$. However, the sign of w plays no role, as reflected in the characterization (2), where w is squared. By reversing its sign, we just obtain the complementary segment of the same conic [5].

3.2 Ellipse or hyperbola

The case $w \neq 1$ yields a central conic, i. e., an ellipse (w < 1) or hyperbola (w > 1). Equation (2) is quadratic in **F** and, after straightforward manipulation, can be written in monic form:

$$\mathbf{F}^2 - 2\mathbf{CF} + \mathbf{d} = 0, \tag{3}$$

with coefficients **C**,**d** expressible as barycentric combinations:

$$\begin{aligned} \mathbf{d} &= (1-\alpha)\mathbf{b}_1^2 + \alpha\mathbf{b}_0\mathbf{b}_2, & \alpha &= 1/(1-w^2), \\ \mathbf{C} &= (1-\alpha)\mathbf{b}_1 + \alpha\mathbf{M}, & \mathbf{M} &= \tfrac{1}{2}(\mathbf{b}_0 + \mathbf{b}_2). \end{aligned} \tag{4}$$

This quadratic equation (3) has two distinct solutions, namely the foci:

$$\mathbf{F} = \mathbf{C} \pm \mathbf{c}, \qquad \mathbf{c} = \sqrt{\mathbf{C}^2 - \mathbf{d}}\,, \tag{5}$$

where the symbol $\sqrt{\ }$ denotes the principal square root of a complex number. Both **C** and **c** (5) admit an immediate geometric interpretation (fig. 5):

- **C** is the midpoint of the segment joining the two foci $\mathbf{F}_-$, $\mathbf{F}_+$, and hence the centre of the conic.
- **c** defines the direction of the major axis, joining **C** and $\mathbf{F}_-$, $\mathbf{F}_+$. The modulus $c = |\mathbf{c}|$ is thus the distance between **C** and either focus, called *linear eccentricity*.

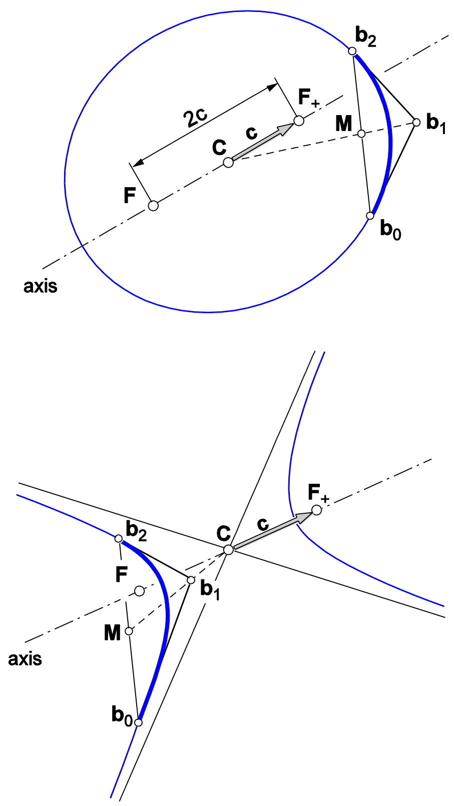

Fig. 5** Foci* **F, centre* **C** *and axis of a Bézier conic: central conics (ellipse and hyperbola)*

Observe that **C** is a barycentric combination (4) of $\mathbf{b}_1$ and the midpoint **M** of the chord $\mathbf{b}_0\mathbf{b}_2$. Therefore, **C** lies on the line joining $\mathbf{b}_1$ and **M**, as already Lee noted [10]. In case **c**=0, we have a circle, where the two foci coalesce ($\mathbf{F}_- = \mathbf{F}_+ = \mathbf{C}$).

3.3 Parabola

A parabola can be regarded as a limiting case of a central conic such that $w = 1$. Therefore, $\alpha = \infty$ for the barycentric coordinate (4), and **C** lies hence at infinity, along the line joining $\mathbf{b}_1$ and **M**. This line defines the axis direction **a** (fig. 6):

$$\mathbf{a} = \mathbf{M} - \mathbf{b}_1, \qquad \mathbf{M} = \tfrac{1}{2}(\mathbf{b}_0 + \mathbf{b}_2)\,. \tag{6}$$

In this case, relationship (2) simplifies to a linear equation in **F**, whose solution is the sole focus of a parabola:

$$\mathbf{F} = \frac{\mathbf{b}_0\mathbf{b}_2 - \mathbf{b}_1^2}{2\mathbf{a}}\,. \tag{7}$$

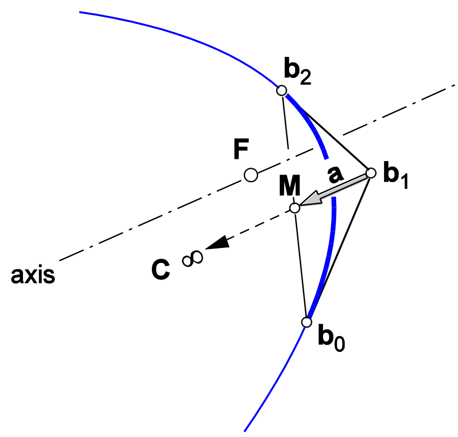

***Fig. 6** Foci* **F** *and axis of a parabola in Bézier form*

4 Other conic characteristics

Once the foci, centre and linear eccentricity are known, the computation of other geometric characteristics, such as axis length, eccentricity or focal parameter, is a straightforward exercise via traditional geometry [3], as explained in this section. We may also use the intermediate *shoulder point* **S** [9] on the Bézier conic, shown in fig. 4:

$$\mathbf{S} = (1-\sigma)\mathbf{b}_1 + \sigma\mathbf{M}, \qquad \sigma = \frac{1}{w+1}\,. \tag{8}$$

4.1 Ellipse or hyperbola

According to its well-known *string construction* [2], an ellipse (or hyperbola) is the locus of points such that the sum (or difference) of their distances to the foci $\mathbf{F}_-$, $\mathbf{F}_+$ is a constant $2a$. This constant is easily calculated from any point **b** on the conic, such as $\mathbf{b}_0$ or $\mathbf{b}_2$, or the shoulder point **S** (8), for the sake of symmetry:

$$2a = |\mathbf{b} - \mathbf{F}_+| \pm |\mathbf{b} - \mathbf{F}_-|, \qquad \begin{array}{l} +\text{: ellipse} \\ -\text{: hyperbola} \end{array}\,.$$

This value coincides with the length $2a$ of the major axis, joining the two vertices. From this length, and the linear eccentricity c, we derive the eccentricity e, as well as the semi-length b of the minor axis (ellipse):

$$e = \frac{c}{a}, \qquad b = \sqrt{c^2 - a^2}\,.$$

4.2 Parabola

A parabola is the locus of points that are equidistant from a point (the focus **F**) and a line (directrix). It corresponds to limit case of an ellipse with unit eccentricity ($e = 1$), and infinite linear eccentricity.

Now the relevant characteristic is the focal parameter p, namely the distance between the focus and the directrix (fig. 7). Similarly to the case of the axis length for an ellipse or hyperbola, p is easily calculated from any point **b** on the conic. We compute now its distance r to the directrix, which equals that to **F** (8), minus the projection of **b**–**F** along the axis, defined by a (7). If β denotes the angle between a and **b**–**F**, then:

$$p = r(1-\cos\beta), \qquad r = |\mathbf{b}-\mathbf{F}|\,.$$

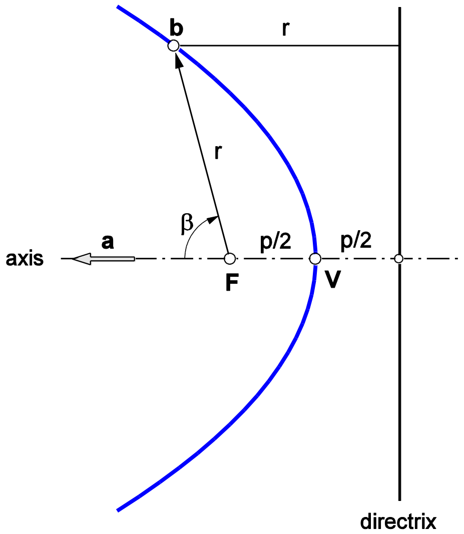

Fig. 7 Foci* F*, centre* C *and axis of a Bézier conic: parabola.

5 Conclusion

Complex arithmetic operations allow a simple, more accessible computation of the characteristics of conic sections in quadratic Bézier form. A complex quadratic equation yields as solutions the foci of a given central conic. Moreover, in the resulting expressions we readily identify the centre, linear eccentricity are direction of the major axis. Thus, the reflective properties of the conic are easily determined. For a parabola, the equation characterizing the focus reduces to a linear one. This approach for obtaining conic characteristics is conceptually and mathematically simpler than existing techniques. Otherwise complicated formulae with scalar and vector products reduce to trivial expressions.

Once the foci, centre and linear eccentricity are known, the computation of other geometric characteristics is a simple exercise.

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